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STABILITY ANALYSIS OF SECOND-ORDER SWITCHED HOMOGENEOUS SYSTEMS*

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Abstract. We study the stability of second-order switched homogeneous systems. Using the concept of *generalized first integrals* we explicitly characterize the "most destabilizing" switching-law and construct a Lyapunov function that yields an easily verifiable, necessary and sufficient condition for asymptotic stability. Using the duality between stability analysis and control synthesis, this also leads to a novel algorithm for designing a stabilizing switching controller.

Key words. absolute stability, switched linear systems, robust stability, hybrid systems, hybrid control

AMS subject classifications. 37N35, 93D15, 93D20, 93D30

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1. Introduction.

We consider the *switched homogeneous system*

$$\dot{\mathbf{x}}(t) \in \mathbf{\Omega}(\mathbf{x}(t)), \quad \mathbf{\Omega}(\mathbf{x}) := Co\{\mathbf{f}_1(\mathbf{x}), \mathbf{f}_2(\mathbf{x}), \dots, \mathbf{f}_q(\mathbf{x})\}, \tag{1.1}$$

where $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))^T$, the $\mathbf{f}_i(\cdot)$'s are homogeneous functions (with equal degree of homogeneity), and Co denotes the convex hull. An important special case is $\mathbf{f}_i(\mathbf{x}) = A_i\mathbf{x}$, $i = 1, \dots, q$, for which (1.1) reduces to a *switched linear system.*

Switched systems appear in many fields of science ranging from economics to electrical and mechanical engineering [15], [18]. In particular, switched linear systems were studied in the literature under various names, e.g., polytopic linear differential inclusions [4], linear polysystems [6], bilinear systems [5], and uncertain linear systems [20].

If $\mathbf{f}_i(\mathbf{0}) = \mathbf{0}$ for all i, then $\mathbf{0}$ is an equilibrium point of (1.1). Analyzing the stability of this equilibrium point is difficult because the system admits infinitely many solutions for every initial value.[1]

Stability analysis of switched *linear* systems can be traced back to the 1940s since it is closely related to the well-known *absolute stability problem* [4], [19]. Current approaches to stability analysis include (i) deriving sufficient but *not* necessary and sufficient stability conditions, and (ii) deriving necessary and sufficient stability conditions for the particular case of low-order systems. Popov's criterion, the circle criterion [19, Chapter 5], and the positive-real lemma [4, Chapter 2] can all be considered as examples of the first approach. Many other sufficient conditions exist in the literature.[2] Nevertheless, these conditions are sufficient but not necessary and sufficient and are known to be rather conservative conditions.

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[1] An analysis of the computational complexity of some closely related problems can be found in [2].

[2] See, for example, the recent survey paper by Liberzon and Morse [11].

Far more general results were derived for the second approach, namely, the particular case of low-order linear switched systems. The basic idea is to single out the "most unstable" solution $\tilde{\mathbf{x}}(t)$ of (1.1), that is, a solution with the following property: If $\tilde{\mathbf{x}}(t)$ converges to the origin, then so do *all* the solutions of (1.1). Then, all that is left to analyze is the stability of this single solution (see, e.g., [3]).

Pyatnitskiy and Rapoport [16] and Rapoport [17] were the first to formulate the problem of finding the "most unstable" solution of (1.1) using a variational approach. Applying the maximum principle, they developed a characterization of this solution in terms of a two-point boundary value problem. Their characterization is not explicit but, nevertheless, using tools from convex analysis they proved the following result. Let Γ be the collection of all the q-sets of *linear* functions $\{A_1\mathbf{x}, \dots, A_q\mathbf{x}\}$ for which (1.1) is asymptotically stable, and denote the boundary[3] of Γ by $\partial\Gamma$. Pyatnitskiy and Rapoport proved that if $\{A_1\mathbf{x}, \dots, A_q\mathbf{x}\} \in \partial\Gamma$, then the "most unstable" solution of (1.1) is a *closed* trajectory. Intuitively, this can be explained as follows. If $\{A\mathbf{x}, B\mathbf{x}\} \in \Gamma$, then, by the definition of Γ, $\tilde{\mathbf{x}}(t)$ converges to the origin; if $\{A\mathbf{x}, B\mathbf{x}\} \notin (\Gamma \cup \overline{\Gamma})$, then $\tilde{\mathbf{x}}(t)$ is unbounded. Between these two extremes, that is, when $\{A\mathbf{x}, B\mathbf{x}\} \in \partial\Gamma$, $\tilde{\mathbf{x}}(t)$ is a closed solution. This leads to a *necessary and sufficient* stability condition for second- and third-order switched linear systems [16], [17]; however, the condition is a nonlinear equation in several unknowns and, since solving this equation turns out to be difficult, it cannot be used in practice.

Margaliot and Langholz [14] introduced the novel concept of *generalized first integrals* and used it to provide a different characterization of the closed trajectory. Unlike Pyatnitskiy and Rapoport, the characterization is *constructive* and leads, for second-order switched linear systems, to an *easily verifiable*, necessary and sufficient stability condition. Furthermore, their approach yields an *explicit* Lyapunov function for switched linear systems.

In the general homogeneous case, the functions $\mathbf{f}_i(\cdot)$ are *nonlinear* functions, and therefore the approaches used for switched linear systems cannot be applied. Filippov [7] derived a necessary and sufficient stability condition for second-order switched homogeneous systems. However, his proof uses a Lyapunov function that is not constructed explicitly.

In this paper we combine Filippov's approach with the approach developed by Margaliot and Langholz to provide a necessary and sufficient condition for asymptotic stability of second-order switched homogeneous systems. We construct a suitable *explicit* Lyapunov function and derive a condition that is easy to check in practice.

A closely related problem is the stabilization of several unstable systems using switching. This problem has recently regained interest with the discovery that there are systems that can be stabilized by *hybrid controllers* whereas they cannot be stabilized by continuous state-feedback [18, Chapter 6]. To analyze the stability of (1.1), we synthesize the "most unstable" solution $\tilde{\mathbf{x}}(t)$ by switching between several asymptotically stable systems. Designing a switching controller is equivalent to synthesizing the "most stable" solution by switching between several unstable systems. These problems are dual and, therefore, a solution of the first is also a solution of the second. Consequently, we use our stability analysis to develop a novel procedure for designing a stabilizing switching controller for second-order homogeneous systems.

The rest of this paper is organized as follows. Section 2 includes some notations and assumptions. Section 3 develops the *generalized first integral* which will serve as our main analysis tool. Section 4 analyzes the sets Γ and $\partial\Gamma$. Section 5 provides an

[3] The set Γ is open [17].

explicit characterization of the "most destabilizing" switching-law. Section 6 presents an easily verifiable, *necessary and sufficient* stability condition. Section 7 describes a new algorithm for designing a switching controller. Section 8 summarizes.

2. Notations and assumptions. For $\beta > 1$, let

$$P_\beta := \left\{ f(\cdot,\cdot) : f(cx_1, cx_2) = c^\beta f(x_1, x_2) \text{ for all } c, x_1, x_2 \right\},$$

that is, the set of homogeneous functions of degree β. We denote by E_β the set of functions $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathbf{f}(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))^T$ with $f_1, f_2 \in P_\beta$.

Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{f} \in E_\beta$. Transforming to polar coordinates,

$$r(t) = \sqrt{x_1^2(t) + x_2^2(t)}, \qquad \theta(t) = \arctan\left(\frac{x_2(t)}{x_1(t)}\right),$$

we get

$$\dot{r} = r^\beta R(\theta), \qquad \dot{\theta} = r^{\beta-1} A(\theta), \tag{2.1}$$

where $R(\theta)$ and $A(\theta)$ are homogeneous functions of degree $\beta+1$ in the variables $\cos(\theta)$ and $\sin(\theta)$.

Following [9, Chapter III], we analyze the stability of (2.1) by considering two cases. If $A(\cdot)$ has no zeros, then the origin is a focus and (2.1) yields $r(\theta) = r_0 e^{\int_{\theta_0}^{\theta} \frac{R(u)}{A(u)} du} = r_0 p(\theta; \theta_0) e^{h\theta}$, where p is periodic in θ with period 2π, and $h := \frac{1}{2\pi} \int_0^{2\pi} \frac{R(u)}{A(u)} du$. Hence, $r(t) \to 0$ ($r(t) \to \infty$) if $sgn(h) \neq sgn(A)$ ($sgn(h) = sgn(A)$).

If A has zeros, say $A(\overline{\theta}) = 0$, then the line $\theta = \overline{\theta}$ is a solution of (2.1) (the origin is a node) and along this line $r(t) \to 0$ ($r(t) \to \infty$) if $R(\overline{\theta}) < 0$ ($R(\overline{\theta}) > 0$).

Hence, if $ES_\beta := \{\mathbf{f} \in E_\beta : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \text{ is asymptotically stable}\}$, then $ES_\beta = ES_\beta^F \cup ES_\beta^N$,[4] where

$$ES_\beta^F := \left\{ \mathbf{f} \in E_\beta : A(\theta) \text{ has no zeros and } sgn(h) \neq sgn(A) \right\},$$
$$ES_\beta^N := \left\{ \mathbf{f} \in E_\beta : R(\overline{\theta}) < 0 \text{ for all } \overline{\theta} \text{ such that } A(\overline{\theta}) = 0 \right\}.$$

Given $\mathbf{f}(\mathbf{x}) \in E_\beta$, we denote its differential at $\mathbf{x}$ by

$$(D\mathbf{f})(\mathbf{x}) := \begin{pmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} \end{pmatrix}.$$

The differential's norm is $||(Df)(\mathbf{x})|| := \sup_{\mathbf{h} \in \mathbb{R}^2,\, ||\mathbf{h}||=1} ||(Df)(\mathbf{x})\mathbf{h}||$, where $||\cdot|| : \mathbb{R}^2 \to \mathbb{R}_+$ is some vector norm on $\mathbb{R}^2$. The distance between two functions $\mathbf{f}, \mathbf{g} \in E_\beta$ is defined by [10]

$$d(\mathbf{f}, \mathbf{g}) := \sup_{\mathbf{x} : ||\mathbf{x}|| < 1} (||\mathbf{f}(\mathbf{x}) - \mathbf{g}(\mathbf{x})|| + ||(D\mathbf{f})(\mathbf{x}) - (D\mathbf{g})(\mathbf{x})||). \tag{2.2}$$

Note that $(E_\beta, d(\cdot,\cdot))$ is a Banach space and that in the topology induced by $d(\cdot,\cdot)$ the set ES_β is *open*.

[4] Here F stands for focus and N for node.

For simplicity,[5] we consider the differential inclusion (1.1) with $q = 2$:

$$\dot{\mathbf{x}}(t) \in \mathbf{\Omega}(\mathbf{x}(t)), \quad \mathbf{\Omega}(\mathbf{x}) := Co\{\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})\} \tag{2.3}$$

with $\mathbf{f}, \mathbf{g} \in ES_\beta$.

Given an initial condition $\mathbf{x}_0$, a solution of (2.3) is an absolutely continuous function $\mathbf{x}(t)$, with $\mathbf{x}(0) = \mathbf{x}_0$, that satisfies (2.3) for almost all t. Clearly, there is an infinite number of solutions for any initial condition. To differentiate the possible solutions we use the concept of a switching-law.

DEFINITION 2.1. *A* switching-law *is a piecewise constant function* $\eta : [0, +\infty) \to [0, 1]$. *We refer to the solution of* $\dot{\mathbf{x}} = \eta(t)\mathbf{f}(\mathbf{x}) + (1 - \eta(t))\mathbf{g}(\mathbf{x})$ *as the solution corresponding to the switching-law* η.

The solution $\mathbf{x}(t) \equiv \mathbf{0}$ is said to be *uniformly*[6] *locally asymptotically stable* if

- given any $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that *every* solution of (2.3) with $||\mathbf{x}(0)|| < \delta(\epsilon)$ satisfies $||\mathbf{x}(t)|| < \epsilon$ for all $t \geq 0$,
- there exists $c > 0$ such that *every* solution of (2.3) satisfies $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$ if $||\mathbf{x}(0)|| < c$.

Since $\mathbf{f}$ and $\mathbf{g}$ are homogeneous, local asymptotic stability of (2.3) implies global asymptotic stability. Hence, when the above conditions hold, the system is uniformly globally asymptotically stable (UGAS).

DEFINITION 2.2. *A set* $P \subset \mathbb{R}^2$ *is an* invariant set *of* (2.3) *if* every *solution* $\mathbf{x}(t)$, *with* $\mathbf{x}(0) \in P$, *satisfies* $\mathbf{x}(t) \in P$ *for all* $t \geq 0$.

DEFINITION 2.3. *We will say that* $\mathbf{\Omega}(\mathbf{x}) = Co\{\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})\}$ *is* singular *if there exists an invariant set that does not contain an open neighborhood of the origin.*

We assume the following from here on.

Assumption 1. The set $\mathbf{\Omega}(\mathbf{x})$ is not singular.

The role of Assumption 1 will become clear in the proof of Lemma 5.4 below. Note that it is easy to check if the assumption holds by transforming the two systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$ to polar coordinates and examining the set of points where $\dot{\theta} = 0$ for each system. For example, if there exists a line l that is an invariant set for both $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$, then l is an invariant set of (2.3) and Assumption 1 does not hold.

To make the stability analysis nontrivial, we also assume the following.

Assumption 2. For any *fixed* $\overline{\eta} \in [0, 1]$, the origin is a globally asymptotically stable equilibrium point of $\dot{\mathbf{x}} = \overline{\eta}\mathbf{f}(\mathbf{x}) + (1 - \overline{\eta})\mathbf{g}(\mathbf{x})$.

3. The generalized first integral.

If the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{3.1}$$

is Hamiltonian [8], then it admits a *classical* first integral, that is, a function $H(\mathbf{x})$ which satisfies $H(\mathbf{x}(t)) \equiv H(\mathbf{x}(0))$ along the trajectories of (3.1). In this case, the study of (3.1) is greatly simplified since its trajectories are nothing but the contours $H(\mathbf{x}) = \text{const}$. In particular, it turns out that the first integral provides a crucial analysis tool for switched linear systems [14]. The purpose of this section is to extend this idea to the case where $\mathbf{f} \in ES_\beta$ and, therefore, (3.1) is not Hamiltonian.

Let $v := x_2/x_1$; then

$$\frac{dv}{\frac{dx_2}{dx_1} - v} = \frac{dx_1}{x_1}.$$

[5] Our results can be easily generalized to the case $q > 2$.

[6] The term "uniform" is used here to describe uniformity with respect to switching signals.

If $\mathbf{f} \in ES_\beta$, then f_1 and f_2 are both homogeneous functions of degree β and, therefore, the ratio $\frac{f_2(x_1,x_2)}{f_1(x_1,x_2)}$ is a function of v only, which we denote by $\alpha(v)$. Hence, along the trajectories of (3.1), $\frac{dv}{\alpha(v)-v} = \frac{dx_1}{x_1}$, that is, $e^{\int \frac{dv}{v-\alpha(v)}}|x_1| = \text{const}$. Thus, we define the *generalized first integral* of (3.1) by

$$H(x_1, v) := \left(x_1 e^{L(v)}\right)^{2k}, \tag{3.2}$$

where $L(v) := \int \frac{dv}{v-\alpha(v)}$ and k is a positive integer. Note that we can write $H = H(x_1, x_2)$ by substituting $v = x_2/x_1$. Note also that $H(\lambda x_1, \lambda x_2) = \lambda^{2k} H(x_1, x_2)$.

Let S be the collection of points where $H(x_1, x_2)$ is not defined or not continuous; then, by construction, $H : \mathbb{R}^2 \setminus S \to \mathbb{R}_+$ is *piecewise constant* along the trajectories of (3.1). If $S = \emptyset$, then H is a classical first integral of the system. In general, however, $S \neq \emptyset$. Nevertheless, this does not imply that H cannot be used in the analysis of (3.1). Consider, for example, the case where S is a line and a trajectory $\mathbf{x}(t)$ of (3.1) can cross S but not stay on S. Then, $H(\mathbf{x}(t))$ will remain constant except perhaps at a crossing time where its value can "jump."[7] Thus, a trajectory of the system is a concatenation of several contours of H. This motivates the term *generalized first integral.*

To clarify the relationship between the trajectories of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and the contours $H(\mathbf{x}) = \text{const}$, we consider an example.

Example 1. *Consider the system*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -x_2^3 - 2x_1^3 \\ x_1 x_2^2 \end{pmatrix}. \tag{3.3}$$

Here (3.2) *yields*

$$H(x_1, v) = \left(x_1 \frac{v(2 - v + v^2)^{\frac{1}{8}}}{(1+v)^{\frac{1}{4}}} e^{-\frac{3}{4\sqrt{7}} \arctan((-1+2v)/\sqrt{7})}\right)^{2k},$$

and using $k = 2$ and $v = x_2/x_1$ we get

$$H(x_1, x_2) = \frac{x_2^4 \sqrt{2x_1^2 - x_1 x_2 + x_2^2}}{x_1 + x_2} e^{-\frac{3}{\sqrt{7}} \arctan(\frac{2x_2 - x_1}{\sqrt{7} x_1})}.$$

In this case $S = l_1 \cup l_2$, where $l_1 := \{\mathbf{x} : x_1 + x_2 = 0\}$ and $l_2 := \{\mathbf{x} : x_1 = 0\}$. It is easy to verify that l_1 is an invariant set of (3.3)*, that is, $\mathbf{x}(t) \cap l_1 = \emptyset$ (except for the trivial trajectory that starts and stays on l_1). Furthermore, it is easy to see that a trajectory of* (3.3) *cannot stay on the line l_2.*

Figure 1 *shows the trajectory $\mathbf{x}(t)$ of* (3.3) *for $\mathbf{x}_0 = (3, 1)^T$. Figure* 2 *displays $H(\mathbf{x}(t))$ as a function of time. It may be seen that $H(\mathbf{x}(t))$ is a piecewise constant function that attains two values. Note that the "jump" in $H(\mathbf{x}(t))$ occurs when $x_1(t) = 0$, that is, when $\mathbf{x}(t) \in S$.*

4. The boundary of stability. Let Γ be the set of all pairs $(\mathbf{f}, \mathbf{g})$ for which (2.3) is UGAS. In this section we study Γ and its boundary $\partial\Gamma$. Our first result, whose proof is given in the appendix, is an inverse Lyapunov theorem.

Lemma 4.1. *If $(\mathbf{f}, \mathbf{g}) \in \Gamma$, then there exists a C^1 positive-definite function $V(\mathbf{x}) : \mathbb{R}^2 \to [0, +\infty)$ such that for all $\mathbf{x} \in \mathbb{R}^2 \setminus \mathbf{0}$, $\nabla V(\mathbf{x})\mathbf{f}(\mathbf{x}) < 0$ and $\nabla V(\mathbf{x})\mathbf{g}(\mathbf{x}) < 0$. Furthermore, $V(\mathbf{x})$ is positively homogeneous of degree one.*[8]

[7] That is, a time t_0 such that $\mathbf{x}(t_0) \in S$.

[8] That is, $V(c\mathbf{x}) = cV(\mathbf{x})$ for all $c > 0$ and all $\mathbf{x} \in \mathbb{R}^2$.

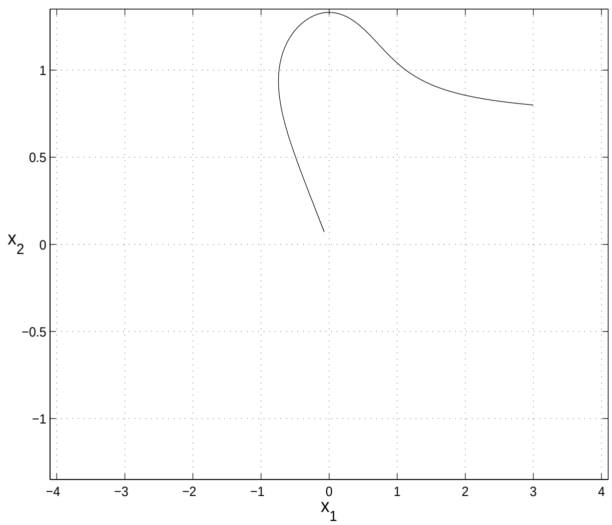

FIG. 1. *The trajectory of* (3.3) *for* $\mathbf{x}_0 = (3,1)^T$.

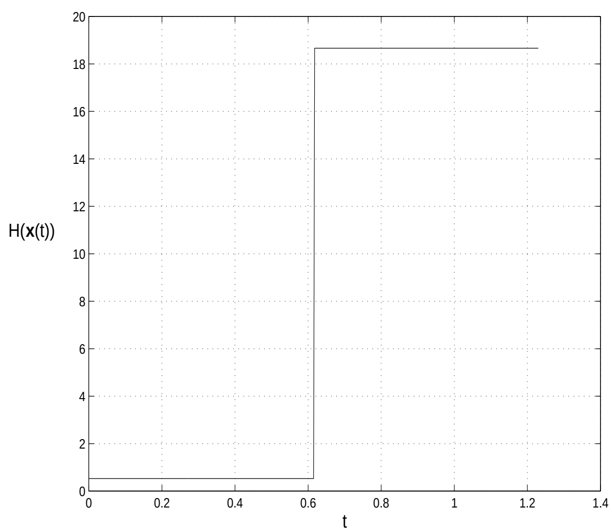

FIG. 2. $H(\mathbf{x}(t))$ *as a function of time.*

LEMMA 4.2. Γ *is an open cone.*

Proof. Let $(\mathbf{f}, \mathbf{g}) \in \Gamma$. Clearly, $(c\mathbf{f}, c\mathbf{g}) \in \Gamma$ for all $c > 0$. Hence, Γ is a cone.

To prove that Γ is open, we use the common Lyapunov function V from Lemma 4.1. Denote $\gamma := \{\mathbf{x} : V(\mathbf{x}) = 1\}$, so γ is a closed curve encircling the origin. Hence, there exists $a < 0$ such that for all $\mathbf{x} \in \gamma$,

$$\nabla V(\mathbf{x})\mathbf{f}(\mathbf{x}) < a \quad \text{and} \quad \nabla V(\mathbf{x})\mathbf{g}(\mathbf{x}) < a. \tag{4.1}$$

If $\tilde{\mathbf{f}} \in ES_\beta$ and $\tilde{\mathbf{g}} \in ES_\beta$ are such that $d(\tilde{\mathbf{f}}, \mathbf{f}) < \epsilon$ and $d(\tilde{\mathbf{g}}, \mathbf{g}) < \epsilon$, with $\epsilon > 0$ sufficiently small, then for all $\mathbf{x} \in \gamma$, $\nabla V(\mathbf{x})\tilde{\mathbf{f}}(\mathbf{x}) < a/2 < 0$ and $\nabla V(\mathbf{x})\tilde{\mathbf{g}}(\mathbf{x}) < a/2 < 0$. It follows from the homogeneity of V, $\tilde{\mathbf{f}}$, and $\tilde{\mathbf{g}}$ that $(\tilde{\mathbf{f}}, \tilde{\mathbf{g}}) \in \Gamma$. ∎

5. The worst-case switching-law. In this section we provide two explicit characterizations of the switching-law that yields the "most unstable" solution of (2.3).

Let $H^{\mathbf{f}}(\mathbf{x})$ ($H^{\mathbf{g}}(\mathbf{x})$) be the generalized first integral of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ ($\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$).

DEFINITION 5.1. *Define the* worst-case switching-law (WCSL) *by*

$$\lambda(\mathbf{x}) := \begin{cases} 0 & \text{if } \nabla H^{\mathbf{f}}(\mathbf{x})\mathbf{g}(\mathbf{x}) \geq 0, \\ 1 & \text{if } \nabla H^{\mathbf{f}}(\mathbf{x})\mathbf{g}(\mathbf{x}) < 0. \end{cases} \tag{5.1}$$

We denote

$$\mathbf{h}(\mathbf{x}) := \lambda(\mathbf{x})\mathbf{f}(\mathbf{x}) + (1 - \lambda(\mathbf{x}))\mathbf{g}(\mathbf{x})$$

so the solution corresponding to the WCSL satisfies $\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x})$. Note that the WCSL is a *state-dependent* switching-law and that since $\lambda(\mathbf{x}) = 0$ or $\lambda(\mathbf{x}) = 1$, then $\mathbf{h}(\mathbf{x}) = \mathbf{g}(\mathbf{x})$ or $\mathbf{h}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$, respectively, that is, the vertices of $\mathbf{\Omega}$. Furthermore, it is easy to see that $\mathbf{h}(\mathbf{x})$ is homogeneous of degree β.

Intuitively, the WCSL can be explained as follows. Consider a point $\mathbf{x}$ where $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are as shown in Figure 3. A solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ follows the contour $H^{\mathbf{f}}(\mathbf{x}) = \text{const}$, whereas a solution of $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$ crosses this contour going further away from the origin. In this case, $\nabla H^{\mathbf{f}}(\mathbf{x})\mathbf{g}(\mathbf{x}) > 0$, so the WCSL is $\lambda(\mathbf{x}) = 0$, which corresponds to setting $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$. Thus, the WCSL "pushes" the trajectory away from the origin as much as possible.

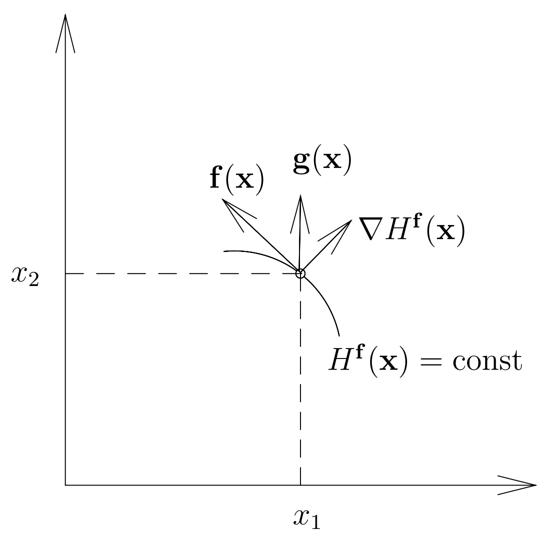

FIG. 3. *Geometrical explanation of the WCSL when* $\nabla H^{\mathbf{f}}(\mathbf{x})\mathbf{g}(\mathbf{x}) > 0$.

Note that the definition of WCSL using (5.1) is meaningful only for $\mathbf{x} \in \mathbb{R}^2 \setminus S$ since $\nabla H^{\mathbf{f}}(\mathbf{x})$ is not defined for $\mathbf{x} \in S$. However, extending the definition of WCSL to any $\mathbf{x} \in \mathbb{R}^2$ is immediate since $\mathbf{x} \in S$ implies one of two cases. In the first case, $\mathbf{x} \in l$, where l is a line in $\mathbb{R}^2$ which is an invariant set of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, that is, $\mathbf{f}(\mathbf{x}) = c\mathbf{x}$ (with $c < 0$ since $\mathbf{f}$ is asymptotically stable), so clearly the WCSL must use $\mathbf{g}$. In the second case, the trajectory of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ crosses S so the value of the switching-law on the single point $\mathbf{x}$ can be chosen arbitrarily.

We expect the WCSL to remain unchanged if we swap the roles of $\mathbf{f}$ and $\mathbf{g}$. Indeed, this is guaranteed by the following lemma, whose proof is given in the appendix.

LEMMA 5.2. *For all* $\mathbf{x} \in D := \{\mathbf{x} : \mathbf{f}^T(\mathbf{x})\mathbf{g}(\mathbf{x}) > 0\}$

$$sgn(\nabla H^{\mathbf{f}}(\mathbf{x})\mathbf{g}(\mathbf{x})) = -sgn(\nabla H^{\mathbf{g}}(\mathbf{x})\mathbf{f}(\mathbf{x})), \tag{5.2}$$

where $sgn(\cdot)$ *is the sign function.*

We can now state the main result of this section.

THEOREM 5.3. $(\mathbf{f}, \mathbf{g}) \in \partial\Gamma$ *if and only if the solution corresponding to the WCSL is closed.*[9]

Proof. Denote the solution corresponding to the WCSL by $\mathbf{x}(t)$ and suppose that $\mathbf{x}(t)$ is closed. Let γ be the closed curve $\{\mathbf{x}(t) : t \in [0, T]\}$, where $T > 0$ is the smallest time such that $\mathbf{x}(T) = \mathbf{x}(0)$. Note that using the explicit construction of $\lambda(\mathbf{x})$ (see (5.1)) we can easily define γ *explicitly* as a concatenation of several contours of $H^{\mathbf{f}}(\mathbf{x})$ and $H^{\mathbf{g}}(\mathbf{x})$. Note also that the switching between $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$ takes place at points $\mathbf{x}$ where $\nabla H^f(\mathbf{x})\mathbf{g}(\mathbf{x}) = 0$ (see (5.1)), that is, when $\mathbf{g}(\mathbf{x})$ and $\mathbf{f}(\mathbf{x})$ are collinear. Hence, the curve γ has no corners.

We define the function $V(\mathbf{x})$ by $V(\mathbf{0}) = 0$, and for all $\mathbf{x} \neq \mathbf{0}$

$$V(\mathbf{x}) = k \quad \text{such that} \quad \mathbf{x} \in k\gamma, \tag{5.3}$$

that is, the contours of V are obtained by scaling γ (see [1]). The function $V(\mathbf{x})$ is positively homogeneous (that is, for any $c \geq 0$, $V(c\mathbf{x}) = cV(\mathbf{x})$), radially unbounded, and differentiable on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.

Let $\mathbf{p}(\mathbf{x}) = ||\mathbf{x}||^{\beta-1}\mathbf{x}$ and denote $\mathbf{f}^\epsilon(\mathbf{x}) := \mathbf{f}(\mathbf{x}) + \epsilon\mathbf{p}(\mathbf{x})$ and $\mathbf{g}^\epsilon(\mathbf{x}) := \mathbf{g}(\mathbf{x}) + \epsilon\mathbf{p}(\mathbf{x})$. Note that both $\mathbf{f}^\epsilon(\mathbf{x})$ and $\mathbf{g}^\epsilon(\mathbf{x})$ belong to E_β. We use $V(\mathbf{x})$ to analyze the stability of the perturbed system $\dot{\mathbf{x}} \in \mathbf{\Omega}^\epsilon(\mathbf{x}) := Co\{\mathbf{f}^\epsilon(\mathbf{x}), \mathbf{g}^\epsilon(\mathbf{x})\}$. Consider the derivative of V along the trajectories of $\dot{\mathbf{x}} \in \mathbf{\Omega}^\epsilon(\mathbf{x})$:

$$\begin{aligned} \dot{V}(\mathbf{x}) &= \nabla V(\mathbf{x})\,(\eta(t)(\mathbf{f}^\epsilon(\mathbf{x}) + (1-\eta(t))\mathbf{g}^\epsilon(\mathbf{x})) \\ &= \epsilon\nabla V(\mathbf{x})\mathbf{p}(\mathbf{x}) + \eta(t)\nabla V(\mathbf{x})\mathbf{f}(\mathbf{x}) + (1-\eta(t))\nabla V(\mathbf{x})\mathbf{g}(\mathbf{x}), \end{aligned}$$

where $\eta(t) \in [0, 1]$ for all t. If at some $\mathbf{x}$, $V(\mathbf{x})$ corresponds to a contour $H^{\mathbf{f}}(\mathbf{x}) = \text{const}$, then $\nabla V(\mathbf{x})\mathbf{f}(\mathbf{x}) = 0$ and, by the definition of WCSL (see (5.1)), $\nabla V(\mathbf{x})\mathbf{g}(\mathbf{x}) \leq 0$ so $\dot{V}(\mathbf{x}) \leq \epsilon\nabla V(\mathbf{x})\mathbf{p}(\mathbf{x})$. Otherwise, $V(\mathbf{x})$ corresponds to a contour $H^{\mathbf{g}}(\mathbf{x}) = \text{const}$, so $\nabla V(\mathbf{x})\mathbf{g}(\mathbf{x}) = 0$, $\nabla V(\mathbf{x})\mathbf{f}(\mathbf{x}) \leq 0$, and again $\dot{V}(\mathbf{x}) \leq \epsilon\nabla V(\mathbf{x})\mathbf{p}(\mathbf{x})$. Hence, for any $\epsilon < 0$ we have

$$\dot{V}(\mathbf{x}) \leq \epsilon\nabla V(\mathbf{x})\mathbf{p}(\mathbf{x}) = \epsilon||\mathbf{x}||^{\beta-1}\nabla V(\mathbf{x})\mathbf{x} < 0;$$

since this holds for all $\mathbf{x}$ and all $\eta(t) \in [0, 1]$, we get that for $\epsilon < 0$, $\mathbf{\Omega}^\epsilon \in \Gamma$.

On the other hand, for $\epsilon > 0$ and $\eta(t) = \lambda(\mathbf{x}(t))$ we have

$$\dot{V}(\mathbf{x}) = \epsilon\nabla V(\mathbf{x})\mathbf{p}(\mathbf{x}) = \epsilon||\mathbf{x}||^{\beta-1}\nabla V(\mathbf{x})\mathbf{x} > 0;$$

since this holds for all $\mathbf{x}$, $\dot{\mathbf{x}} \in \mathbf{\Omega}^\epsilon(\mathbf{x})$ admits an unbounded solution for $\epsilon > 0$. The derivations above hold for arbitrarily small ϵ and, therefore, $\mathbf{\Omega} \in \partial\Gamma$.

For the opposite direction, assume that $(\mathbf{f}, \mathbf{g}) \in \partial\Gamma$, and let $\mathbf{x}(t)$ be the solution corresponding to the WCSL, that is, $\mathbf{x}(t)$ satisfies $\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x}) := \lambda(\mathbf{x})\mathbf{f}(\mathbf{x}) +$

[9] We omit specifying the initial condition because the fact that $\mathbf{h}(\mathbf{x})$ is homogeneous implies that, if the solution starting at some $\mathbf{x}_0$ is closed, then all solutions are closed.

$(1-\lambda(\mathbf{x}))\mathbf{g}(\mathbf{x})$. To prove that $\mathbf{x}(t)$ is a closed trajectory, we use the following lemma, whose proof appears in the appendix.

LEMMA 5.4. *If* $(\mathbf{f},\mathbf{g}) \in \partial\Gamma$*, then the solution corresponding to the WCSL rotates around the origin.*

Thus, for a given $\mathbf{x}_0 \neq \mathbf{0}$, there exists $t_1 > 0$ such that $\mathbf{x}(t)$, with $\mathbf{x}(0) = \mathbf{x}_0$, satisfies $\mathbf{x}(t_1) = c\mathbf{x}_0$, and since $\mathbf{h}(\mathbf{x})$ is homogeneous, we get $\mathbf{x}(nt_1) = c^n\mathbf{x}(0)$, $n = 1,2,3,\dots$. We consider two cases. If $c > 1$, then $\mathbf{x}(t)$ is unbounded, and using the homogeneity of $\mathbf{h}(\mathbf{x})$ we conclude that $\mathbf{0}$ is a (spiral) source. It follows from the theory of structural stability (see, e.g., [10]) that there exists an $\epsilon > 0$ such that for *all* $(\tilde{\mathbf{f}},\tilde{\mathbf{g}})$ with $d(\tilde{\mathbf{f}},\mathbf{f}) < \epsilon$ and $d(\tilde{\mathbf{g}},\mathbf{g}) < \epsilon$, the origin is a source of the perturbed dynamical system $\dot{\mathbf{x}} = \lambda(x)\tilde{\mathbf{f}}(\mathbf{x}) + (1-\lambda(x))\tilde{\mathbf{g}}(\mathbf{x})$. This implies that $(\mathbf{f},\mathbf{g}) \notin \partial\Gamma$, which is a contradiction.

If $c < 1$, then $\mathbf{x}(t)$ converges to the origin and, by the construction of the WCSL, so does any other solution, so $(\mathbf{f},\mathbf{g}) \in \Gamma$, which is again a contradiction. Hence, $c = 1$, that is, $\mathbf{x}(t)$ is closed. □

The characterization of the WCSL using the generalized first integrals leads to a simple and *constructive* proof of Theorem 5.3. However, to actually check whether the solution corresponding to the WCSL is closed, a characterization of the WCSL in polar coordinates is more suitable.

Representing (2.3) in polar coordinates, we get

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} \in \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} Co\{\mathbf{f}(r,\theta),\mathbf{g}(r,\theta)\}. \tag{5.4}$$

If $(\mathbf{f},\mathbf{g}) \in \partial\Gamma$, then the WCSL yields a closed solution. By using the transformation $\overline{r} = r$, $\overline{\theta} = -\theta$ (if necessary), we may always assume that this solution rotates around the origin in a counterclockwise direction, that is, $\dot{\theta}(r,\theta) > 0$ for all $\theta \in [0,2\pi)$. Note that this implies that if at some point $\mathbf{x}$ the trajectories of one of the systems are in the clockwise direction, then the WCSL will use the second system. Hence, determining the WCSL is nontrivial only at points where the trajectories of both systems rotate in the same direction, and we assume from here on that both rotate in a clockwise direction. (Note that this explains why in Lemma 5.2 it is enough to consider $\mathbf{x} \in D$.)

Let $\mathbf{j}_\eta(r,\theta) := \eta\mathbf{f}(r,\theta) + (1-\eta)\mathbf{g}(r,\theta)$ and

$$F(r,\theta) := \{\eta \in [0,1] : (-\sin\theta \quad \cos\theta)\mathbf{j}_\eta(r,\theta) > 0\} \tag{5.5}$$

so F is a parameterization of the set of directions in $\mathbf{\Omega}$ for which $\dot{\theta} > 0$.

For any (r,θ) we define the switching-law

$$\zeta(r,\theta) := \arg\max_{\eta\in F} \frac{1}{r}\frac{\dot{r}}{\dot{\theta}};$$

that is, ζ is the switching-law that selects, among all the directions which yield $\dot{\theta} > 0$, the direction that maximizes $\frac{d\ln r}{d\theta}$. Using (5.4), we get

$$\zeta(r,\theta) = \arg\max_{\eta\in F} \frac{(\cos\theta \quad \sin\theta)\mathbf{j}_\eta(r,\theta)}{(-\sin\theta \quad \cos\theta)\mathbf{j}_\eta(r,\theta)}. \tag{5.6}$$

Let

$$m(r,\theta) := \frac{(\cos\theta \quad \sin\theta)\mathbf{j}_\zeta(r,\theta)}{(-\sin\theta \quad \cos\theta)\mathbf{j}_\zeta(r,\theta)} \tag{5.7}$$

so that along the trajectory corresponding to ζ, $\frac{1}{r}\dot{r}/\dot{\theta} = m$. Note that since $\mathbf{f}$ and $\mathbf{g}$ are homogeneous, $\zeta = \zeta(\theta)$ and $m = m(\theta)$.

It is easy to verify that the function $q(y) := \frac{ay+b(1-y)}{cy+d(1-y)}$, $y \in [0,1]$ (where c and d are such that the denominator is never zero), is monotonic and, therefore, $\zeta(r,\theta)$ in (5.6) is always 0 or 1 and $m(r,\theta)$ in (5.7) is always one of the two values,

$$m_0(\theta) := \frac{(\cos\theta \quad \sin\theta)\mathbf{g}(r,\theta)}{(-\sin\theta \quad \cos\theta)\mathbf{g}(r,\theta)}, \qquad m_1(\theta) := \frac{(\cos\theta \quad \sin\theta)\mathbf{f}(r,\theta)}{(-\sin\theta \quad \cos\theta)\mathbf{f}(r,\theta)},$$

respectively.

The next lemma, whose proof is given in the appendix, shows that the switching-law ζ is just the WCSL λ.

LEMMA 5.5. *The switching-law ζ yields a closed solution if and only if λ yields a closed solution.*

Let

$$\begin{aligned} I &:= \int_0^{2\pi} m(\theta)d\theta \\ &= \int_0^{2\pi} \frac{d\ln r}{d\theta}d\theta \\ &= \ln(r(T)) - \ln(r(0)), \end{aligned} \tag{5.8}$$

where $(r(t),\theta(t))$ is the solution corresponding to the switching-law ζ, and T is the time needed to complete a rotation around the origin. This solution is closed if and only if $\ln(r(T)) - \ln(r(0)) = 0$. Combining this with Lemma 5.5 and Theorem 5.3, we immediately obtain the following.

THEOREM 5.6. $(\mathbf{f},\mathbf{g}) \in \partial\Gamma$ *if and only if* $I = 0$.

It is easy to calculate I numerically and, therefore, Theorem 5.6 provides us with a simple recipe for determining whether $(\mathbf{f},\mathbf{g}) \in \partial\Gamma$. However, note that we assumed throughout that the closed solution of the system rotates in a counterclockwise direction. Thus, to use Theorem 5.6 correctly, I has to be computed twice: first for the original system and then for the transformed system $r' = r$, $\theta' = -\theta$ (denote this value by I'). $(\mathbf{f},\mathbf{g}) \in \partial\Gamma$ if and only if $\max(I,I') = 0$. In this way, we find whether the system has a closed trajectory, rotating around the origin in a clockwise or counterclockwise direction.

The following example demonstrates the use of Theorem 5.6.

EXAMPLE 2 (detecting the boundary of stability). *Consider the system*

$$\dot{\mathbf{x}} \in \Omega_k(\mathbf{x}) := Co\{\mathbf{f}(\mathbf{x}), \mathbf{g}_k(\mathbf{x})\}, \tag{5.9}$$

where

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_2^3 - 2x_1^3 \\ x_1x_2^2 \end{pmatrix}, \qquad \mathbf{g}_k(\mathbf{x}) = \begin{pmatrix} (kx_1 - x_2)^3 - 2x_1^3 \\ x_1(x_2 - kx_1)^2 \end{pmatrix}. \tag{5.10}$$

It is easy to verify that $\mathbf{f} \in ES_3^N$, and since $\mathbf{g}_0(\mathbf{x}) = \mathbf{f}(\mathbf{x})$, we have $\Omega_0 \in \Gamma$. The problem is to determine the smallest $k^* > 0$ such that $(\mathbf{f}(\mathbf{x}), \mathbf{g}_{k^*}(\mathbf{x})) \in \partial\Gamma$.

Transforming to polar coordinates we get

$$\mathbf{f}(r,\theta) = r^3 \begin{pmatrix} -\sin^3\theta - 2\cos^3\theta \\ \cos\theta\sin^2\theta \end{pmatrix}, \quad \mathbf{g}_k(r,\theta) = r^3 \begin{pmatrix} (k\cos\theta - \sin\theta)^3 - 2\cos^3\theta \\ \cos\theta(\sin\theta - k\cos\theta)^2 \end{pmatrix},$$

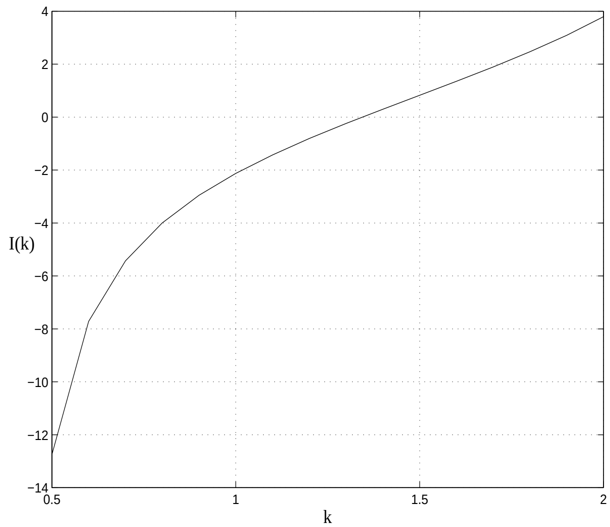

FIG. 4. *$I(k)$ as a function of k.*

so

$$j_\eta(r,\theta) = r^3\eta \begin{pmatrix} -\sin^3\theta - 2\cos^3\theta \\ \cos\theta\sin^2\theta \end{pmatrix} + r^3(1-\eta) \begin{pmatrix} (k\cos\theta - \sin\theta)^3 - 2\cos^3\theta \\ \cos\theta(\sin\theta - k\cos\theta)^2 \end{pmatrix}$$

and

$$I = \int_0^{2\pi} m(\theta)d\theta = \int_0^{2\pi} \max_{\eta\in F(\theta)} \frac{(\cos\theta \quad \sin\theta)\mathbf{j}_\eta(r,\theta)}{(-\sin\theta \quad \cos\theta)\mathbf{j}_\eta(r,\theta)} d\theta, \tag{5.11}$$

where $F(\theta)$ includes 0 if $(-\sin\theta \ \ \cos\theta)\mathbf{j}_0(r,\theta) > 0$ and 1 if $(-\sin\theta \ \ \cos\theta)\mathbf{j}_1(r,\theta) > 0$. Note that although j_η is a function of both r and θ, the integrand in (5.11) is a function of θ (and k) but not of r.

We calculated $I(k)$ numerically for different values of k. The results are shown in Figure 4. The value k^* for which $I(k^*) = 0$ is

$$k^* = 1.3439$$

(to four-digit accuracy), and it may be seen that for $k < k^*$ ($k > k^*$), $I(k) < 0$ ($I(k) > 0$). We repeated the computation for the transformed system $\overline{r} = r$, $\overline{\theta} = -\theta$ and found that there exists no closed solution rotating around the origin in a clockwise direction. Hence, the system (5.9) and (5.10) is UGAS for all $k \in [0, k^*)$ and unstable for all $k > k^*$.

The WCSL (see (5.6)) for $k = k^*$ is

$$\zeta(\theta) = \begin{cases} 0 & \text{if } \theta \in [0, 0.6256) \cup [1.1811, 3.7672) \cup [4.3227, 2\pi), \\ 1 & \text{otherwise.} \end{cases} \tag{5.12}$$

Figure 5 depicts the solution of the system given by (5.9) and (5.10) with $k =$ 1.3439, WCSL (5.12), and $\mathbf{x}_0 = (1,0)^T$. It may be seen that the solution is a closed trajectory, as expected. Note that this trajectory is not *convex, which implies that the Lyapunov function used in the proof of Theorem 5.3 (see (5.3)) is not convex. This is*

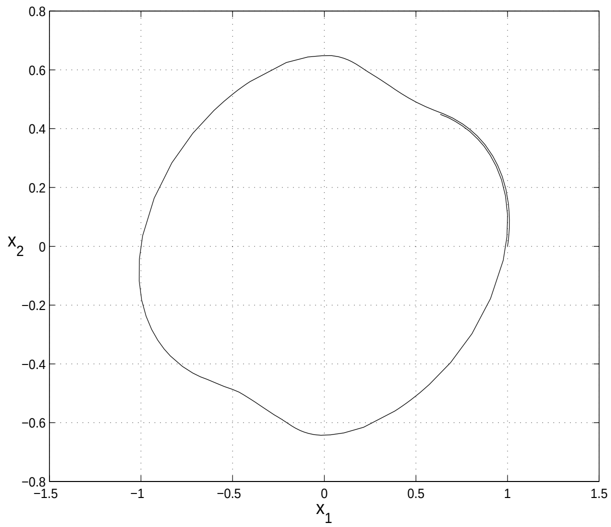

FIG. 5. *The solution of* (5.9) *and* (5.10) *for* $k = k^*$ *and the WCSL, with* $\mathbf{x}_0 = (1, 0)^T$.

a phenomenon that is unique to nonlinear systems. For switched linear *systems the closed trajectory is convex and, therefore, so is the Lyapunov function* V *that yields a sufficient and necessary stability condition* [14].

6. Stability analysis. In this section we transform the original problem of analyzing the stability of (2.3) to one of detecting the boundary of stability $\partial\Gamma$. The latter problem was solved in section 5.

Given $\mathbf{\Omega} = Co\{\mathbf{f}, \mathbf{g}\}$, we define a new homogeneous function $\mathbf{h}_k(\mathbf{x})$ with the following properties: (1) $\mathbf{h}_0(\mathbf{x}) = \mathbf{f}(\mathbf{x})$; (2) $\mathbf{h}_1(\mathbf{x}) = \mathbf{g}(\mathbf{x})$; and (3) for all $k_1 < k_2$, $\{\mathbf{h}_k(\mathbf{x}) : 0 \le k \le k_1\} \subset \{\mathbf{h}_k(\mathbf{x}) : 0 \le k \le k_2\}$. One possible example that satisfies the above is

$$\mathbf{h}_k(\mathbf{x}) := \mathbf{f}(\mathbf{x}) + k(\mathbf{g}(\mathbf{x}) - \mathbf{f}(\mathbf{x})).$$

Consider the switched homogeneous system

$$\dot{\mathbf{x}}(t) \in \mathbf{\Omega}_k(\mathbf{x}(t)), \quad \mathbf{\Omega}_k := Co\{\mathbf{f}(\mathbf{x}), \mathbf{h}_k(\mathbf{x})\}. \tag{6.1}$$

The *absolute stability problem* is to find the smallest $k^* > 0$, when it exists, such that $\mathbf{\Omega}_{k^*} \in \partial\Gamma$. Noting that $\mathbf{\Omega}_0 = Co\{\mathbf{f}(\mathbf{x}), \mathbf{f}(\mathbf{x})\} \in \Gamma$, $\mathbf{\Omega}_1 = Co\{\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})\} = \mathbf{\Omega}$, and $\mathbf{\Omega}_{k_1} \subset \mathbf{\Omega}_{k_2}$ for all $k_1 < k_2$, we immediately obtain the following result.

LEMMA 6.1. *The system* (2.3) *is UGAS if and only if* $k^* > 1$.

Thus, we can always transform the problem of analyzing the stability of a switched dynamical system into an absolute stability problem. We already know how to solve the latter problem for second-order homogeneous systems. To illustrate this consider the following example.

EXAMPLE 3. *Consider the system* (2.3) *with*

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} -x_2^3 - 2x_1^3 \\ x_1 x_2^2 \end{pmatrix}, \qquad \mathbf{g}(\mathbf{x}) = \begin{pmatrix} (x_1 - x_2)^3 - 2x_1^3 \\ x_1(x_2 - x_1)^2 \end{pmatrix}. \tag{6.2}$$

It is easy to verify that $\mathbf{f}(\mathbf{x})$ *and* $\mathbf{g}(\mathbf{x})$ *belong to* ES_3 *and that both Assumptions* 1 *and* 2 *are satisfied.*

To analyze the stability of the system we use Lemma 6.1. *Defining*

$$\mathbf{h}_k(\mathbf{x}) = \begin{pmatrix} (kx_1 - x_2)^3 - 2x_1^3 \\ x_1(x_2 - kx_1)^2 \end{pmatrix}, \tag{6.3}$$

we must find the smallest k^* *such that* $(\mathbf{f}, \mathbf{h}_{k^*}) \in \partial\Gamma$. *We already calculated* k^* *in Example* 2 *and found that* $k^* = 1.3439 > 1$. *Hence, the system* (2.3) *with* $\mathbf{f}$ *and* $\mathbf{g}$ *given in* (6.2) *is UGAS.*

7. Designing a switching controller. In this section we employ our results to derive an algorithm for designing a switching controller for stabilizing homogeneous systems. To be concrete, we focus on linear systems rather than on the general homogeneous case. Hence, consider the system

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \quad \mathbf{u} \in \mathcal{U} := Co\{K_1\mathbf{x}, K_2\mathbf{x}\}, \tag{7.1}$$

where K_1 and K_2 are given matrices that represent constraints on the possible controls.[10] We would like to design a stabilizing state-feedback controller $\mathbf{u}(t) = \mathbf{u}(\mathbf{x}(t))$ that satisfies the constraint $\mathbf{u}(t) \in \mathcal{U}$ for all t.

We assume that for any fixed matrix $K \in Co\{K_1, K_2\}$ the matrix $A + BK$ is strictly unstable and, therefore, a linear controller $\mathbf{u} = K\mathbf{x}$ will not stabilize the system. However, it is still possible that a switching controller will stabilize the system, and designing such a controller (if one exists) is the purpose of this section.

Roughly speaking, we are trying to find a switching-law that yields an asymptotically stable solution of $\dot{\mathbf{x}} \in \mathbf{\Omega} := Co\{A + BK_1, A + BK_2\}\mathbf{x}$, where each matrix in $\mathbf{\Omega}$ is strictly unstable. Using the transformation $\bar{t} = -t$, we see that such a solution exists if and only if this switching-law yields an unstable solution of $\dot{\mathbf{x}} \in \mathbf{\Omega}^- := Co\{-(A + BK_1), -(A + BK_2)\}\mathbf{x}$. Clearly, every matrix in $\mathbf{\Omega}^-$ is asymptotically stable. Hence, we obtain the main result of this section.

THEOREM 7.1. *Let* $\lambda = \lambda(\mathbf{x})$ *be the WCSL for the system*

$$\dot{\mathbf{x}} \in Co\{-(A + BK_1), -(A + BK_2)\}$$

and let $\tilde{\mathbf{x}}$ *be the corresponding solution. There exists a switching controller that asymptotically stabilizes* (7.1) *if and only if* $\tilde{\mathbf{x}}$ *is unbounded and, in this case,* $\mathbf{u}(\mathbf{x}) = \lambda(\mathbf{x})K_1\mathbf{x} + (1 - \lambda(\mathbf{x}))K_2\mathbf{x}$ *is a stabilizing controller.*

Note that Theorem 7.1 provides an algorithm for designing a stabilizing switching controller whenever such a controller exists. We already solved the problem of analyzing $\tilde{\mathbf{x}}$ for second-order systems.

EXAMPLE 4 (designing a stabilizing switching controller). *Consider the system* (7.1) *with*

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \tag{7.2}$$

where $k > 0$ *is a constant. It is easy to verify that for any fixed* $K \in Co\{K_1, K_2\}$, *the matrix* $A + BK$ *is unstable and, therefore, no linear controller* $\mathbf{u} = K\mathbf{x}$ *can stabilize the system. Therefore, we design a switching controller. By Theorem* 7.1 *we must analyze the stability of the switched system* (6.1) *with*

$$\mathbf{f}(\mathbf{x}) = -\begin{pmatrix} 0 & 1 \\ -2 & 1 \end{pmatrix}\mathbf{x}, \qquad \mathbf{h}_k(\mathbf{x}) = -\begin{pmatrix} 0 & 1 \\ -(2+k) & 1 \end{pmatrix}\mathbf{x}.$$

[10]Determined, for example, by the physical limitations of the actuators.

Transforming $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ *to polar coordinates, we get*

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} (\cos\theta - \sin\theta) r \sin\theta \\ (\sin\theta - \frac{1}{2}\cos\theta)^2 + \frac{7}{4}\cos^2\theta \end{pmatrix},$$

whereas $\dot{\mathbf{x}} = \mathbf{h}_k(\mathbf{x})$ *becomes*

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} ((1+k)\cos\theta - \sin\theta) r \sin\theta \\ (\sin\theta - \frac{1}{2}\cos\theta)^2 + (\frac{7}{4} + k)\cos^2\theta \end{pmatrix}.$$

Clearly, the solutions of both these systems always rotate in a counterclockwise direction ($\dot{\theta} > 0$ *for all* θ) *and, therefore, for all* θ, *we have* $m(\theta) = \max(m_0(\theta), m_1(\theta))$, *where*

$$m_0(\theta) = \frac{((1+k)\cos\theta - \sin\theta)\sin\theta}{(\sin\theta - \frac{1}{2}\cos\theta)^2 + (\frac{7}{4} + k)\cos^2\theta}, \qquad m_1(\theta) = \frac{(\cos\theta - \sin\theta)\sin\theta}{(\sin\theta - \frac{1}{2}\cos\theta)^2 + \frac{7}{4}\cos^2\theta}.$$

It is easily verified that $m_1(\theta) \le m_0(\theta)$ *if and only if* $\tan\theta \ge 0$. *Hence, the WCSL is*

$$\zeta(\theta) = \begin{cases} 0 & \text{if } \tan\theta \ge 0, \\ 1 & \text{otherwise} \end{cases} = \begin{cases} 0 & \text{if } \theta \in [0, \pi/2) \cup [\pi, 3\pi/2), \\ 1 & \text{otherwise} \end{cases}$$

and

$$I(k) = \int_0^{\pi/2} m_0(\theta) d\theta + \int_{\pi/2}^{\pi} m_1(\theta) d\theta + \int_{\pi}^{3\pi/2} m_0(\theta) d\theta + \int_{3\pi/2}^{2\pi} m_1(\theta) d\theta.$$

Computing numerically, we find that the value of k *for which* $I = 0$ *is* $k^* = 6.98513$. *Hence, there exists a switching controller that asymptotically stabilizes* (7.1) *and* (7.2) *if and only if* $k > 6.98513$ *and*

$$\mathbf{u}(\mathbf{x}) = \begin{cases} K_2\mathbf{x} & \text{if } \arctan(x_2/x_1) \in [0, \pi/2) \cup [\pi, 3\pi/2), \\ K_1\mathbf{x} & \text{otherwise} \end{cases} \tag{7.3}$$

is a stabilizing controller.

Figure 6 *depicts the trajectory of the closed-loop system given by* (7.1) *and* (7.2) *with* $k = 10$, *the switching controller* (7.3), *and* $\mathbf{x}_0 = (1, 0)^T$. *As we can see, the system is indeed asymptotically stable.*

8. Summary. We presented a new approach to stability analysis of second-order switched homogeneous systems based on the idea of generalized first integrals. Our approach leads to an explicit Lyapunov function that provides an easily verifiable, *necessary and sufficient* stability condition.

Using our stability analysis, we designed a novel algorithm for constructing a *switching controller* for stabilizing second-order homogeneous systems. The algorithm determines whether the system can be stabilized using switching, and if the answer is affirmative, outputs a suitable controller.

Interesting directions for further research include the complete characterization of the boundary of stability $\partial\Gamma$ and the study of higher-order switched homogeneous systems.

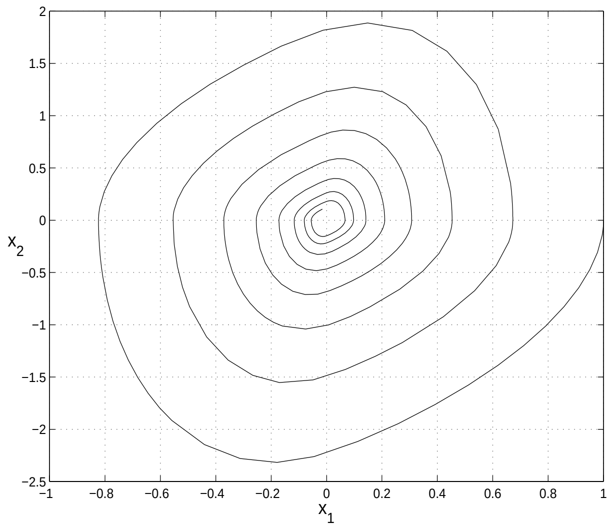

FIG. 6. *Trajectory of the closed-loop system with the switching controller with* $\mathbf{x}_0 = (1, 0)^T$.

Appendix.

Proof of Lemma 4.1. The existence of a common Lyapunov function $V'(\mathbf{x})$ follows from Theorem 3.1 in [13] (see also [12]). However, V' is not necessarily homogeneous. Denote $\gamma := \{\mathbf{x} : V'(\mathbf{x}) = 1\}$ so γ is a closed curve encircling the origin. We define a new function $V(\mathbf{x})$ by $V(\mathbf{0}) = 0$ and, for all $\mathbf{x} \neq \mathbf{0}$,

$$V(\mathbf{x}) = k \quad \text{such that} \quad \mathbf{x} \in k\gamma;$$

that is, the contours of V are obtained by scaling γ (see [1]). $V(\mathbf{x})$ is differentiable on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, positively homogeneous of order one, and radially unbounded.

For any $\mathbf{x} \in \gamma$ we have $\nabla V(\mathbf{x})\mathbf{f}(\mathbf{x}) = \nabla V'(\mathbf{x})\mathbf{f}(\mathbf{x}) < 0$, and using the homogeneity of $V(\mathbf{x})$ and $\mathbf{f}(\mathbf{x})$ this holds for any $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Similarly, $\nabla V(\mathbf{x})\mathbf{g}(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. ∎

Proof of Lemma 5.2. Let $\mathbf{v}(\mathbf{x}) = \frac{\mathbf{f}(\mathbf{x})}{||\mathbf{f}(\mathbf{x})||}$ and $\mathbf{w}(\mathbf{x}) = \frac{(\nabla H^{\mathbf{f}}(\mathbf{x}))^T}{||\nabla H^{\mathbf{f}}(\mathbf{x})||}$. These two vectors form an orthonormal basis of $\mathbb{R}^2$ and, therefore, $\mathbf{g}(\mathbf{x}) = a_1\mathbf{v}(\mathbf{x}) + a_2\mathbf{w}(\mathbf{x})$ and $(\nabla H^{\mathbf{g}}(\mathbf{x}))^T = b_1\mathbf{v}(\mathbf{x}) + b_2\mathbf{w}(\mathbf{x})$, where $a_1 = \mathbf{g}^T(\mathbf{x})\mathbf{v}(\mathbf{x})$, $a_2 = \mathbf{g}^T(\mathbf{x})\mathbf{w}(\mathbf{x})$, $b_1 = \nabla H^{\mathbf{g}}(\mathbf{x})\mathbf{v}(\mathbf{x})$, and $b_2 = \nabla H^{\mathbf{g}}(\mathbf{x})\mathbf{w}(\mathbf{x})$. Now $\nabla H^{\mathbf{g}}(\mathbf{x})\mathbf{g}\mathbf{x} = 0$ yields

$$a_1 b_1 + a_2 b_2 = 0. \tag{8.1}$$

For any $\mathbf{x} \in D$ we have $a_1 > 0$ and since $\nabla H^{\mathbf{f}}(\mathbf{x})$ $(\nabla H^{\mathbf{g}}(\mathbf{x}))$ is orthogonal to $\mathbf{f}(\mathbf{x})$ $(\mathbf{g}(\mathbf{x}))$, we also have $b_2 > 0$. Substituting in (8.1) yields $sgn(a_2) = -sgn(b_1)$, which is just (5.2). ∎

Proof of Lemma 5.4. The system $\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x})$ is homogeneous and we can represent it in polar coordinates as in (2.1). If $A(\overline{\theta}) = 0$ for some $\overline{\theta} \in [0, 2\pi]$, then the solution corresponding to the WCSL follows the line $l := \theta = \overline{\theta}$. If $R(\overline{\theta}) < 0$, then the solution follows the line l to the origin. However, by the definition of WCSL this is possible only if both the solutions of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$ coincide with the line l. Thus, the line l is an invariant set of the system which is a contradiction to

Assumption 1. If $R(\overline{\theta}) \geq 0$, then we get a contradiction of Assumption 2. Hence, $A(\theta) \neq 0$ for all $\theta \in [0, 2\pi]$ and, therefore, there exists $c > 0$ such that $A(\theta) > c$ or $A(\theta) < -c$ for all $\theta \in [0, 2\pi]$. Thus, the solution rotates around the origin. □

Proof of Lemma 5.5. Suppose that the WCSL yields a closed trajectory $\tilde{\mathbf{x}}(t)$ that rotates around the origin in a counterclockwise direction ($\dot{\theta} > 0$). Assume that at some point $\mathbf{x}$ along this trajectory, $\lambda(\mathbf{x}) = 1$, that is,

$$\nabla H^{\mathbf{f}}(\mathbf{x})\mathbf{g}(\mathbf{x}) < 0. \tag{8.2}$$

Note that by the definition of the generalized first integral, $\nabla H^{\mathbf{f}}(\mathbf{x})\mathbf{f}(\mathbf{x}) = 0$ for any $\mathbf{x} \in \mathbb{R}^2 \setminus S$. This implies that $\nabla H^{\mathbf{f}}(\mathbf{x}) = k(f_2(\mathbf{x}), -f_1(\mathbf{x}))$ for some $k > 0$, so (8.2) yields

$$f_2(\mathbf{x})g_1(\mathbf{x}) - f_1(\mathbf{x})g_2(\mathbf{x}) < 0. \tag{8.3}$$

Let r, θ be the polar coordinates of $\mathbf{x}$. Since $\tilde{\mathbf{x}}(t)$ rotates around the origin in a counterclockwise direction and satisfies $\dot{\tilde{\mathbf{x}}} = \mathbf{f}(\tilde{\mathbf{x}})$ at $\mathbf{x}$, we have $(-\sin\theta \quad \cos\theta)\mathbf{f}(r, \theta) > 0$. If $(-\sin\theta \quad \cos\theta)\mathbf{g}(r, \theta) < 0$, then $0 \notin F(r, \theta)$ and, therefore, $\zeta(\theta) = 1$. If, on the other hand, $(-\sin\theta \quad \cos\theta)\mathbf{g}(r, \theta) > 0$, then by the definition of ζ (see (5.6)), $\zeta(\theta) = 1$ if and only if

$$\frac{(\cos\theta \quad \sin\theta)\mathbf{f}(r,\theta)}{(-\sin\theta \quad \cos\theta)\mathbf{f}(r,\theta)} > \frac{(\cos\theta \quad \sin\theta)\mathbf{g}(r,\theta)}{(-\sin\theta \quad \cos\theta)\mathbf{g}(r,\theta)}. \tag{8.4}$$

Simplifying, we see that (8.4) is equivalent to $f_1(r, \theta)g_2(r, \theta) - f_2(r, \theta)g_1(r, \theta) > 0$, which is just (8.3), hence, $\zeta(r, \theta) = 1$. Summarizing, we proved that $\lambda(\mathbf{x}) = 1$ if and only if $\zeta(\theta) = 1$. □

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