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Singular perturbations and first order PDE on manifolds

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Abstract. In this Note we present some results concerning the concentration of sequences of first eigenfunctions on the limit sets of a Morse–Smale dynamical system on a compact Riemannian manifold. More precisely a renormalized sequence of eigenfunctions converges to a measure μ concentrated on the hyperbolic sets of the field. The coefficients which appear in the limit measure can be characterized using the concentration theory.
In the second part, certain aspects of some first order PDE on manifolds are studied. We study the limit of a sequence of solutions of a second order PDE, when a parameter of viscosity tends to zero. We exhibit the role played by the limit sets of the vector fields. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Perturbations singulières et équations aux dérivées partielles du premier ordre sur des variétés

Résumé. *Dans cette Note, nous présentons des résultats de concentration d'une suite de fonctions propres associées à la première valeur propre sur les ensembles limites d'un système de type Morse–Smale, l'étude est réalisée sur une variété riemannienne compacte. Plus précisément, une suite renormalisée de fonctions propres converge vers une mesure μ concentrée sur les ensembles hyperboliques du champ de vecteurs. Les coefficients à la limite sont caractérisés en utilisant la théorie de la concentration.*
Dans une seconde partie, nous étudions certaines propriétés des solutions des EDP du premier ordre sur une variété. La limite d'une suite, solution d'une EDP du second ordre, est déterminée lorsqu'un paramètre de viscosité tend vers zéro. On met en évidence le rôle joué par les ensembles limites du champ de vecteurs. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

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Soit (V_n, g) une variété riemannienne compacte. On s'intéresse ici à la première valeur propre et la première fonction propre de l'opérateur $L_\varepsilon = \varepsilon\Delta + \sum_{i=1}^n b_i\partial_i + a$, avec $\Delta_g = -\nabla_i\nabla^i$, a est une fonction régulière et b est un champ de vecteurs de type Morse–Smale.

Nous avons prouvé le résultat suivant :

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THÉORÈME 1. – *Sur une variété compacte* (V_n, g) *de dimension* $n \geqslant 2$, *soit* Ω *un champ de vecteurs de type Morse–Smale qui n'est pas un gradient et tel que l'ensemble récurrent consiste en l'union de points stationnaires* $P_1, \ldots, P_M$ *et d'orbites périodiques* $\Gamma_1, \ldots, \Gamma_N$. *Soient L une fonction de Lyapunov associée à* Ω *et* $b = \Omega - \nabla L$.

Pour $\varepsilon > 0$, *considérons* λ_ε, u_ε, *respectivement la première valeur propre et la fonction propre associée à l'opérateur*

$$\varepsilon \Delta_g + b\nabla + a \quad \textit{sur } V_n.$$

Alors l'ensemble des limites faibles quand ε *tend vers zéro, de la famille normalisée de mesures*

$$\frac{\mathrm{e}^{-L/\varepsilon} u_\varepsilon^2 \,\mathrm{d}V_g}{\int_{V_n} \mathrm{e}^{-L/\varepsilon} u_\varepsilon^2 \,\mathrm{d}V_g}$$

est contenué de mesures ν *de la forme* $\nu = \sum_{j=1}^N \mu_j + \sum_{i=1}^M c_i^2 \delta_{P_i}$, *où chaque* μ_j *est une mesure supportée par un cycle* Γ_j *de* b, δ_{P_i} *est un Dirac en un point stationnaire* P_i *et les* c_i^2 *sont des constantes positives.*

L'approche de perturbations singulières permet aussi d'obtenir des résultats d'existence d'EDP linéaires et quasi linéaires sur les variétés riemanniennes et surtout de comprendre comment la géométrie de la variété détermine la solution.

La méthode de viscosité a été très utilisée pour prouver des résultats d'existence sur des ouverts à bord. En général, la solution d'une EDP du premier ordre est déterminée par la propagation de la donnée au bord. Sans bord, l'information provient des points critiques du champ. Le second résultat s'énonce de la façon suivante :

THÉORÈME 2. – *Sur une variété compacte* (V_n, g), *on considère un champ de vecteurs de type Morse–Smale* b *et soient* c *une fonction positive vérifiant* $c(x) \geqslant c_0 > 0$ *et* f *une fonction différentiable. Sous ces hypothèses, l'EDP du premier ordre* :

$$\langle b, \nabla u \rangle + cu = f \quad \textit{sur } V_n$$

admet une solution $u \in \mathrm{C}^0(V_n)$ *telle que* $|\nabla u| \in \mathrm{L}_\infty(V_n)$. *De plus, si l'ensemble récurrent associé au champ de vecteurs* $b(x)$ *consiste en l'union d'un nombre fini de points* $P_1, \ldots, P_p$, *et si* c_0 *est une constante suffisamment grande devant les valeurs propres de* $\mathrm{D}b$, *alors* u *est l'unique solution* $\mathrm{W}^{1,\infty}$, *appartenant à* $\mathrm{C}^1(V_n - S)$ *et est complètement déterminée par les valeurs* $u(P_i) = f(P_i)/c(P_i)$.

Quand le champ b *possède des cycles limites, la solution n'est pas unique et la limite de* u *le long des trajectoires convergentes vers un cycle limite n'existe pas.*

Dans le cas quasi linéaire, sous certaines hypothèses, on obtient des résultats du même type pour les solutions de

$$\big\langle b(u,x), \nabla u \big\rangle + c(u,x)u = f \quad \text{sur } V_n.$$

1. Introduction

Let (V_n, g) denote a compact Riemannian manifold of dimension $n \geqslant 2$, with no boundary and $\Delta_g = -\nabla_i \nabla^i$ be the Laplace–Beltrami operator [1]. This Note concerns the study of the operator $L_\varepsilon = \varepsilon\Delta + \sum_{i=1}^n b_i \partial_i + a$ acting on smooth functions, when the parameter ε converges to zero; b is a regular vector field and a is a positive function.

In the first part, we study the behavior of the first eigenfunction u_ε associated to the the smallest eigenvalue $\lambda_\varepsilon > 0$ of the operator $L_\varepsilon(u_\varepsilon) = \lambda_\varepsilon u_\varepsilon$, when the parameter ε converges to zero. A nonlinear version of this problem, when the field is a gradient can be found in [7].

Some local results about the λ_ε and u_ε sequences are known on bounded domains of $\mathbb{R}^n$ (*see* Friedmann [2,5,3] Friedlin–Ventcell [4]: when b has only one attracting point, u_ε converges uniformly on every compact set to a constant as ε converges to 0, but when the point is repulsive it converges in the distribution sense to a Dirac distribution centered at this point.

The behavior of the first eigenvalue λ_ε as ε goes to zero is well known for a large class of dynamical systems: λ_ε converges to the topological pressure P. This quantity is characterized by a variational problem on the set of probability measures: when the field b has a finite number of hyperbolic invariant sets K, then the topological pressure is

$$P = \sup\left(h_\mu + \int_{V_n}\left(c - \frac{\mathrm{d}\det \mathrm{D}\phi_t^{\mathrm{u}}}{\mathrm{d}t}\right)\mathrm{d}\mu \,\middle|\, \operatorname{supp}\mu \subset K \text{ and } \mu \text{ is } \phi_t\text{-invariant}\right).$$

The set of measures considered here is all the measures with support in K and invariant by the flow. h_μ is the metric entropy (*see* [11], ϕ_t is the flow induced by the vector field b and $\mathrm{D}\phi_t^{\mathrm{u}}$ is the differential of ϕ restricted to the unstable bundle of K.

The variational problem associate to the topological pressure P is attained by a measure μ, called the equilibrium state (*see* [11,17]). When the recurrence set of the field b consists of p hyperbolic sets K_i, $i = 1,\dots,p$, the measure μ is concentrated on the union of the K_i's. More precisely, $\mu = \sum_{i=1}^{p} p_i \mu_{K_i}$ where μ_{K_i} is the equilibrium state associated with the set K_i, $p_i \geqslant 0$ and $\sum_{i=1}^{p} p_i = 1$ (*see* Theorem 3.4 in [10]).

Unfortunately, this fruitful approach is not suitable for the study of the first eigenfunction problem since the equilibrium measures are not invariant by the flow, in general. However, we shall see that the limit measures have their supports on the recurrent sets of the field b, although they are not invariant by the flow.

In the second part, we give some results about the behavior of the solutions of the equation $L_\varepsilon u_\varepsilon = f$ when ε goes to zero, for a given smooth function f. Our main interest is to find bounded C^1-solutions and to understand the interaction between the geometry of the characteristic curves (trajectories) of the field b and the behavior of the solutions of $L_\varepsilon u_\varepsilon = f$ when ε goes to zero. The fascinating point here is that when ε goes to zero the elliptic equation $L_\varepsilon u_\varepsilon = f$ tends to a hyperbolic one. In this last case the singularities of the solutions propagate along the characteristics whereas, in the elliptic cases there is no propagation of singularities.

The fields considered here are Morse–Smale (*see* [13,16]), that is: 1) the recurrent set consists of a finite number of hyperbolic stationary points and periodic orbits, 2) the stable and unstable manifolds of the recurrent orbits are pairwise transversal (for all p, q, the unstable manifold $\mathrm{W^u}(p)$ of p and the stable manifold $\mathrm{W^s}(q)$ of q intersect transversally: $\mathrm{T}(\mathrm{W^u}(p)) + \mathrm{T}(\mathrm{W^s}(q)) = \mathrm{T}(V_n)$). A more general class of vector fields will be considered elsewhere.

2. Singular perturbation for the first eigenfunction

THEOREM 1. – *Suppose that the first eigenvalue of the operator $\Delta_g + a$ is positive, a is a function with a finite number of minimum points which are not degenerate* (*in the sense of Morse*). *Consider the first eigenvalue problem which has the following variational formulation*

$$\lambda_\varepsilon = \inf_{u \in \mathrm{H}_1(V_n)-\{0\}} \frac{\varepsilon \int_{V_n} |\nabla u|^2 + a\,u^2}{\int_{V_n} u^2}.$$

Then, when ε converges to zero, we recall that λ_ε converges to the minimum of the function a and the set of limits for the weak topology, when ε goes to zero, of the family of measures $(u_\varepsilon^2\,\mathrm{d}V_g)/\int_{V_n} u_\varepsilon^2\,\mathrm{d}V_g$ defined by the positive solution u_ε of the PDE,

$$\varepsilon \Delta_g u_\varepsilon + a\,u_\varepsilon = \lambda_\varepsilon u_\varepsilon \quad \text{on } V_n$$

is contained in the simplex $M = \{\nu = \sum_{i=1}^{n} c_i^2 \delta_{P_i}, \sum_{i=1}^{n} c_i^2 = 1\}$ of all probability measures with support in the finite set $\{P_i \mid i = 1,\dots,n\}$, where δ_P denotes the Dirac measure at the point P.

Remark. – u_ε is uniquely defined up to a multiplicative constant by the Krein–Rutman theorem.

Consider now the case when $b=\nabla\phi$ and the function c is chosen so that the eigenvalue λ_ε of the operator $L_\varepsilon=\varepsilon\Delta+\sum_{i=1}^n b_i\partial_i+c$ is positive on the manifold. To study the family u_ε of solutions of the PDE

$$\varepsilon\Delta u_\varepsilon+\sum_{i=1}^n b_i\partial_i u_\varepsilon+c\,u_\varepsilon=\lambda_\varepsilon u_\varepsilon \quad \text{on } V_n, \tag{1}$$

we substitute the new function $v_\varepsilon=u_\varepsilon\psi_\varepsilon$ for u_ε where $\psi_\varepsilon=\mathrm{e}^{-\phi/2\varepsilon}$. Equation (1) is transformed into the following PDE where the vector field disappear:

$$\varepsilon^2\Delta v_\varepsilon+a_\varepsilon v_\varepsilon=\varepsilon\lambda_\varepsilon v_\varepsilon \quad \text{on } V_n,$$

where $a_\varepsilon=c\varepsilon+\frac{|\nabla\phi|^2}{4}+\frac{\varepsilon\Delta\phi}{2}$.

PROPOSITION 1. – *Suppose that the following condition is satisfied: at the critical points P_i of the function ϕ, $c(P_i)+\Delta\phi(P_i)/2\geqslant 0$. Let v_ε be a minimizer of the following variational problem*

$$\varepsilon\lambda_\varepsilon=\inf_{u\in \mathrm{H}_1(V_n)-\{0\}}\frac{\varepsilon^2\int_{V_n}|\nabla u|^2+a_\varepsilon u^2}{\int_{V_n}u^2},$$

then:

- $\lim_{\varepsilon\to 0}\varepsilon\lambda_\varepsilon=\inf_{V_n}(\nabla\phi)^2=0$,
- *the weak limits of the normalized measures* $(\mathrm{e}^{-\phi/\varepsilon}u_\varepsilon^2\,\mathrm{d}V_g)/\int_{V_n}\mathrm{e}^{-\phi/\varepsilon}u_\varepsilon^2$ *have their support in the set of critical points of* ϕ.
  $\sup_{V_n}v_\varepsilon$ *tends to* $+\infty$ *as* ε *goes to zero* (*blow up*).

Next we will consider vector fields of the following form $b=-\nabla L+\Omega$, where Ω is fixed but not of gradient type, and we will construct L to be a Lyapunov function of Ω. Using this vector field, we study the behavior of the first eigenfunction for the operator L_ε and obtain the following theorem:

THEOREM 2. – *On a compact Riemannian manifold V_n, consider a Morse–Smale vector field Ω not a gradient whose recurrent set consists of the stationary points $P_1,\dots,P_M$ and of the periodic orbits $\Gamma_1,\dots,\Gamma_N$. L denotes a special Lyapunov function associated with Ω defined in the proof of Lemma* 2.

For $\varepsilon>0$ let λ_ε, u_ε denote, respectively the first eigenvalue and the associated eigenfunction of the operator

$$\varepsilon\Delta_g+b\nabla+c \quad \textit{on } V_n.$$

Then the weak limits of the family of normalized measures

$$\frac{\mathrm{e}^{-L/\varepsilon}u_\varepsilon^2\,\mathrm{d}V_g}{\int_{V_n}\mathrm{e}^{-L/\varepsilon}u_\varepsilon^2\,\mathrm{d}V_g}$$

are of the form $\nu=\sum_{j=1}^N\mu_j+\sum_{i=1}^M c_i^2\delta_{P_i}$ where μ_j are supported by the limit cycle Γ_j and the P_i's, $1\leqslant i\leqslant N$, are the critical points of b. The c_i^2 are nonnegative scalars and the following normalization condition is satisfied:

$$1=\sum_{j=1}^N\int_{\Gamma_j}\mu_j+\sum_{i=1}^M c_i^2.$$

To prove the theorem, we need three lemmas.

LEMMA 1. – *Given a Morse–Smale field Ω, there exists C^∞ Lyapunov functions for Ω, taking the value zero on all the repulsors of Ω and such that the function $\Psi(L)=\frac{1}{4}(|\nabla L|^2+2(\nabla L,\Omega))$ is positive except on the recurrent set of Ω where it is zero.*

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A similar type of Lyapunov function L was constructed by Kamin [8,9] in the case of an attractive point.

LEMMA 2. – *Under the assumptions of Theorem* 2 *on the vector field* Ω,

$$\lim_{\varepsilon\to 0} \varepsilon\lambda_\varepsilon = 0 = \min_{V_n} \Psi,$$

where $\Psi = \frac{(\nabla L)^2}{4} + \frac{(\Omega, \nabla L)}{2}$.

LEMMA 3. – *Under the assumptions of Theorem* 2 *on the vector field* Ω, *all weak limits of measures* $v_\varepsilon^2 \, \mathrm{d}V_g$ *as* ε *goes to zero are concentrated on the minimum set of the function* Ψ.

3. First order PDE on manifolds

We study the limit of the solution u_ε of $L_\varepsilon(u_\varepsilon) = f$, where f is a given smooth function on the manifold, when ε tends to zero. The limit of the sequence when ε goes to zero, solves some first order PDE. For some previous works see [14,15] and [12]. c and f are two given positive smooth functions and b a vector field. We suppose that $c_0 = \inf_{V_n} c > 0$ and $b_0 = \sup_{X\in \mathrm{T}V_n} 1/2 \sum_{i,k} (\nabla_i b_k + \nabla_k b_i) X^i X^k$ is a finite number. S denotes the set of separatrices associates to the dynamical system. We have the following theorem:

THEOREM 3. – *On a compact Riemannian manifold, consider a Morse–Smale vector field* b *and let* c *be a positive function satisfying* $c(x) \geqslant c_0 > 0$ *and* $c_0 - b_0 > 0$. f *is a differentiable function. Under these assumptions, the first order PDE*:

$$\langle b, \nabla u\rangle + c\,u = f \quad \text{on } V_n$$

admits a solution $u \in \mathrm{C}^0(V_n)$ *such that* $|\nabla u| \in \mathrm{L}_\infty(V_n)$. *Moreover, if the recurrent set of the vector field* $b(x)$ *consists of a finite number of points* $P_1, \ldots, P_p$, *and* c_0 *is a constant larger than the eigenvalues of* $\mathrm{D}b$, *then* u *is the unique solution, belongs to* $\mathrm{C}^1(V_n - S) \cap \mathrm{W}^{1,\infty}$ *and is completely determined by the values* $u(P_i) = f(P_i)/c(P_i)$. *When* b *possess some limit cycles, the solution is not unique and has no limit near the limit cycles.*

We adapt the previous theorem to the following nonlinear first order PDE on a compact manifold:

$$\big\langle b(u,x), \nabla u\big\rangle + c(u,x)u = f, \tag{2}$$

where $b(\lambda, x)$ is a regular vector field, λ is parameter and $c(\lambda, x)$ and f are two given functions. The purpose of this part is to find some conditions on f, c and b to insure the existence of regular solutions, since there exists examples where shocks occur. To prove the existence of solutions for (2), we use an elliptic regularization and proceed by successive approximations (*see* Jausselin et al. [6]). For our last theorem we need the following notations:

$$b_0 = \frac{1}{2} \sup_{\|X\|=1,\, x\in V_n,\, \lambda\in\mathbb{R}} \sum_{i,k} \big(\nabla_i b_k(x,\lambda) + \nabla_k b_i(x,\lambda)\big) X^i X^k$$

and $\gamma = \sup_{\lambda\in\mathbb{R},\, x\in V_n} |\partial_\lambda b(\lambda, x)|$, $a_0 = \inf_{\lambda,x} c(\lambda,x) - b_0$. $A = \sup_{V_n} |\nabla f| + \sup_{V_n}(f/c) \times \sup_{V_n\times\mathbb{R}} |\partial c/\partial x|$ and $\beta = \sup_{V_n\times\mathbb{R}} |c'| \times \sup_{V_n}(f/c)$, where we use the notation $c' = \partial c/\partial\lambda$.

We make the following assumptions for the rest of this section:

(i) $a_0 > \beta$ that is $\inf_{\lambda,x} c(\lambda,x) - b_0 > \sup_{V_n} |c'| \sup_{V_n}(f/c)$;

(ii) if $\Lambda = a_0 - \beta$, then $\Lambda^2 - 4A\gamma \geqslant 0$.

THEOREM 4. – *On a compact manifold, consider the parametrized smooth vector field* $b(\lambda, x)$. *Let* $c(\lambda, x)$ *satisfies* $c(\lambda,x) \geqslant c_0 > 0$, *where* c_0 *large so that* $c_0(c_0 - b_0) > \sup_{V_n}(f|c'|)$ *and suppose that the conditions* (i) *and* (ii) *above are satisfied. Then the first order PDE*:

$$\big\langle b(u,x), \nabla u\big\rangle + c(u,x)u = f \quad \text{on } V_n$$

admits a solution $u \in \mathrm{W}^{1,\infty}(V_n)$.

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